Combinatorics in Banach space theory

PROBLEMS (Part 3)*

● **PROBLEM 3.1.** Verify that Phillips' lemma may be equivalently stated by saying that the canonical projection $\pi\colon c_0^{***}\to c_0^*$ is sequentially weak*-to-norm continuous. By the canonical projection from X^{***} onto X^* we mean the one given by $\pi(x^{***})=x^{***}|_{j(X)}$, where $j\colon X\to X^{**}$ is the canonical embedding.

Remark. It is easy to see that $\pi=j^*$. This is the so-called *Dixmier projection* which was to be found in Problem 1.1(a).

● **PROBLEM 3.2.** We say that a Banach space X has the [*weak*] *Phillips property* whenever the Dixmier projection from X^{***} onto X^* is sequentially weak*-to-norm [weak*-to-weak] continuous. Show that for every Banach space X with the weak Phillips property the dual X^* is weakly sequentially complete (that is, every weakly Cauchy sequence is weakly convergent).

Remark. In view of this assertion, we may say that Phillips' lemma implies that ℓ_1 is weakly sequentially complete.

● **PROBLEM 3.3.** Let $(e_n)_{n=1}^\infty$ be the sequence of canonical unit vectors in c_0. Define a vector measure $\mu\colon\mathcal{P}\mathbb{N}\to c_0$ by the formula $\mu(A)=\sum_{n\in A}\frac{1}{n}e_n$ for $A\subset\mathbb{N}$. Determine the semivariation $\|\mu\|$.

● **PROBLEM 3.4.** Let μ be a real-valued, σ-additive measure defined on a σ-algebra Σ. Let also $P\in\Sigma$ be a positive set from the Hahn decomposition theorem, that is, a set satisfying $\mu(A\cap P)\geqslant 0$ and $\mu(A\setminus P)\leqslant 0$ for every $A\in\Sigma$, and let $\mu^+(A)=\mu(A\cap P)$ and $\mu^-(A)=-\mu(A\setminus P)$ $(A\in\Sigma)$. Show that $|\mu|=\mu^++\mu^-$.

● **PROBLEM 3.5.** Let λ and μ be two σ-additive measures defined on a σ-algebra Σ, where λ is complex-valued and μ is non-negative and σ-finite. Suppose $\lambda\ll\mu$, i.e. $\mu(E)=0$ implies $\lambda(E)=0$, for every $E\in\Sigma$. Show that $|\lambda|\ll\mu$.

● **PROBLEM 3.6.** Let μ be a finite, σ-additive, non-negative measure defined on a σ-algebra Σ, and let $\mathscr{F}\subset\Sigma$ be a set algebra. Assume that $(f_n)_{n=1}^\infty$ is a bounded and equi-integrable sequence of functions from $L_1(\mu)$ (see Definition 3.4) such that the limit $F(E)=\lim_{n\to\infty}\int_E f_n\,\mathrm{d}\mu$ exists for each $E\in\mathscr{F}$. Prove that this limit exists also for every E belonging to Σ', the σ-algebra generated by $\mathscr{F}$, and it defines a σ-additive measure $F\colon\Sigma'\to\mathbb{R}$.

● **PROBLEM 3.7.** Let K be a compact, Hausdorff space and $(\mu_n)_{n=1}^\infty\subset\mathcal{M}(K)$ be a sequence of regular, scalar-valued, σ-additive Borel measures on K such that for every sequence $(U_n)_{n=1}^\infty$ of pairwise disjoint open subsets of K we have $\lim_{n\to\infty}\mu_n(U_n)=0$. Show that for every such sequence $(U_n)_{n=1}^\infty$ we have also $\lim_{n\to\infty}|\mu_n|(U_n)=0$.

● **PROBLEM 3.8.** Let K and $\mathcal{M}(K)$ be like in the previous problem. For any non-negative measure $\mu\in\mathcal{M}(K)$ let $\mathcal{AC}(\mu)$ be the set of all measures from $\mathcal{M}(K)$ that are absolutely

*Evaluation: ● =2pt, ● =3pt, ● =4pt

continuous with respect to μ. Prove that $\mathcal{AC}(\mu)$ is a closed subspace of $\mathcal{M}(K)$ (recall that the norm considered here is the total variation norm).

● **PROBLEM 3.9.** Several equivalent clauses may be accepted as a definition of Grothendieck space, but the most common is the following one: A Banach space X is called a *Grothendieck space* whenever every w^*-null sequence in X^* is weakly null.

Suppose X is a Grothendieck space and Y is a Banach space such that there is a surjective bounded linear operator $T\colon X \to Y$. Directly from the above definition show that Y is also a Grothendieck space. In other words, the property of being a Grothendieck spaces is inherited by quotients.

Hint. Use the fact that T^* is weak*-to-weak* continuous and that T^{**} maps X^{**} onto Y^{**}.

● **PROBLEM 3.10.** Show that every separable quotient of ℓ_∞ is reflexive.

Hint. Use Problem 3.9.

Remark. By H.P. Rosenthal's result from 1968 (see Theorem ___ from the lecture notes), we know that the most fundamental infinite-dimensional reflexive space, that is ℓ_2, actually is a quotient of ℓ_∞ and, even more, $\ell_2(\mathfrak{c})$ is also a quotient of ℓ_∞. This is a great application of big independent families!

● **PROBLEM 3.11.** Show that, unlike ℓ_1, the space $L_1(0,1)$ does not have Schur's property.

Hint. Try some orthonormal sequence from $L_2(0,1)$.

● **PROBLEM 3.12.** Disprove that ℓ_1 is a Grothendieck space by pointing to a concrete example of a w^*-null sequence in ℓ_∞ ($\simeq \ell_1^*$) that is not weakly null.

Hint. You will need some special functional from ℓ_∞ to show that your sequence is not weakly null, namely, the *Banach limit* (in fact, there are many of them). See, e.g., Exercise III.4 in [Rud91].

● **PROBLEM 3.13.** Let K be a compact, Hausdorff space and Σ be the σ-algebra of all Borel subsets of K. Let also $\mathcal{A} \subset \mathcal{M}(K)$ be a family of scalar-valued, σ-additive measures defined on Σ. Show that the following three statements are equivalent:

(i) $\mathcal{A}$ is uniformly regular (see Definition 3.3);
(ii) $\mathcal{A}$ is *uniformly σ-additive*, that is, for every decreasing sequence $(E_n)_{n=1}^\infty \subset \Sigma$ we have $\lim_{n\to\infty} \sup_{\mu\in\mathcal{A}} |\mu|(E_n) = 0$;
(iii) $\mathcal{A}$ is both *outer* and *inner* uniformly regular, that is, for every $E \in \Sigma$ and every $\varepsilon > 0$ there exists an open set $V \supset E$ and a compact set $H \subset E$ such that $\sup_{\mu\in\mathcal{A}} |\mu|(V \setminus H) < \varepsilon$.

● **PROBLEM 3.14.** Let $\mathcal{F}$ be a bounded subset of $L_1(\mu)$, where μ is a σ-finite, σ-additive, non-negative measure defined on some σ-algebra. Show that the following two conditions are equivalent:

(i) $\mathcal{F}$ is equi-integrable (see Definition 3.4);
(ii) $\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f| \,\mathrm{d}\mu = 0$.

Hint. For the implication (i) $\Rightarrow$ (ii) use the Chebyshev inequality: $\mu(\{|f| > M\}) \leqslant \|f\|_1/M$, for any $f \in L_1(\mu)$ and any $M > 0$.

● **PROBLEM 3.15.** Show that there exists a functional $\varphi \in \ell_\infty^*$ such that for each $x \in \ell_\infty$ the value $\varphi(x)$ is equal to one of the partial limits of the sequence x.

Hint. Incorporate the Banach–Alaoglu theorem.

● **PROBLEM 3.16.** Let K and $\mathcal{M}(K)$ be like in Problem 3.7 and let $\lambda, \mu \in \mathcal{M}(K)$, $\mu \geqslant 0$, satisfy $\lambda \ll \mu$. Let also $f \in L_1(\mu)$ be the Radon–Nikodým derivative of λ with respect to μ ($f = \mathrm{d}\lambda/\mathrm{d}\mu$). Prove that

$$|\lambda|(E) = \int_E |f| \, \mathrm{d}\mu \quad \text{for every Borel set } E \subset K.$$

● **PROBLEM 3.17.** For each $n \in \mathbb{N}$ let e_n^* be the nth coordinate functional defined on c_0. Do there exist any Hahn–Banach extensions f_n^* of e_n^*, defined on the space ℓ_∞, with the property that for every $x \in \ell_\infty$ the series $\sum_{n=1}^\infty f_n^*(x) e_n$ converges in norm (not necessarily to x)?